Jackson

4-6 (a) We wish to compute $\sum_i \sum_j Q_{ij} \frac{\partial E_j}{\partial x_i} \Big|_0$.

$$= Q_{11} \frac{\partial E_1}{\partial x_1} + Q_{12} \frac{\partial E_2}{\partial x_1} + Q_{13} \frac{\partial E_3}{\partial x_1}$$

$$+ Q_{21} \frac{\partial E_1}{\partial x_2} + Q_{22} \frac{\partial E_2}{\partial x_2} + Q_{23} \frac{\partial E_3}{\partial x_2} \Bigg|_0$$

$$+ Q_{31} \frac{\partial E_1}{\partial x_3} + Q_{32} \frac{\partial E_2}{\partial x_3} + Q_{33} \frac{\partial E_3}{\partial x_3}$$

For a quadrupole of a nucleus, the only nonvanishing component of q_{lm} is $q_{20} = (\cdots) Q_{33}$.

$\Rightarrow$ $q_{21} = 0$, $q_{22} = 0$.

Recall $q_{l,-m} = (-1)^m q^*_{lm}$

$\Rightarrow$ $q_{2-1} = 0$, $q_{2-2} = 0$.

$Q_{12} \propto q_{22} - q_{2-2} = 0$,

$Q_{13} \propto q_{2-1} + q_{21} = 0$,

$Q_{23} \propto q_{2-1} - q_{21} = 0$.

By definition, it's clear that Q is symmetric

$$\Rightarrow \quad Q = \begin{bmatrix} -\frac{1}{2} Q_{33} & 0 & 0 \\ 0 & -\frac{1}{2} Q_{33} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}.$$

$$\sum_i \sum_j Q_{ij} \frac{\partial E_j}{\partial x_i}\bigg|_0 = \left[Q_{33} \frac{\partial E_z}{\partial z}\bigg|_0 - \frac{1}{2} Q_{33} \frac{\partial E_x}{\partial x}\bigg|_0 - \frac{1}{2} Q_{33} \frac{\partial E_y}{\partial y}\bigg|_0 \right].$$

$$= Q_{33} \left[\frac{\partial E_z}{\partial z}\bigg|_0 - \frac{1}{2}\left(\frac{\partial E_x}{\partial x}\bigg|_0 + \frac{\partial E_y}{\partial y}\bigg|_0 \right) \right].$$

$$= \frac{3}{2} Q_{33} \frac{\partial E_z}{\partial z}\bigg|_0.$$

$$\Rightarrow \quad W = -\frac{e}{6} \sum_i \sum_j Q_{ij} \frac{\partial E_j}{\partial x_i}\bigg|_0$$

$$= -\frac{e}{6} \frac{3}{2} Q_{33} \frac{\partial E_z}{\partial z}\bigg|_0$$

$$= \boxed{-\frac{e}{4} Q \frac{\partial E_z}{\partial z}\bigg|_0}$$

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Jackson

4.6 (b) First compute $\left.\frac{\partial E_z}{\partial z}\right|_0$ in a conventional unit:

$$\left.\frac{\partial E_z}{\partial z}\right|_0 = -\frac{4}{e}\frac{W}{Q} = -\frac{4}{e}\frac{h \times 10^7\ s^{-1}\ Js}{2 \times 10^{-28}\ m^2}$$

$$= \boxed{-\frac{4 \times h \times 10^7}{2 \times 10^{-28}}\ \frac{J}{e\,m^2}}$$

The above computation is in units of $\frac{\text{Joules}}{(\text{charge})(\text{meter})^2}$

$$\frac{e}{4\pi\varepsilon_0 a_0^3} = \frac{e}{4\pi\varepsilon_0 a_0^3}\ \frac{1}{m^3}$$ in units, where a_0^3 is now taken as a pure number.

$\frac{e}{4\pi\varepsilon_0}\frac{1}{m}$ is the potential of 1 ~~Coulomb~~ electron at 1 meter, which is 1.44×10^{-9} V by Jackson section 1.2.

$$\Rightarrow \frac{e}{4\pi\varepsilon_0 a_0^3} = \cancel{\frac{e}{4\pi\varepsilon_0 a_0^3}}(1.44 \times 10^{-9})\ \frac{V}{m^2}$$, again, the rightside expression has extracted units out of a_0.

$$1eV \approx 1.6 \times 10^{-19}\ J \Rightarrow \boxed{\frac{V}{m^2} = 1.6 \times 10^{-19}\ \frac{J}{em^2}}$$

This is great! Because The two boxed terms have same dimension!

$$\frac{e}{4\pi\varepsilon_0 a_0^3} = 1.44 \times 10^{-9} \times 1.6 \times 10^{-19}\ \frac{J}{em^2} = 2.3 \times 10^{-28}\ \frac{J}{em^2}.$$

$$\left.\frac{\partial E_z}{\partial z}\right|_0 = \frac{-4 \times h \times 10^7}{2 \times 10^{-28}}\ \frac{J}{em^2}\left[2.3 \times 10^{-28}\ \frac{J}{em^2}\right]^{-1} 2.3 \times 10^{-28}\ \frac{J}{em^2}$$

$$= \frac{-4 \times h \times 10^7}{2 \times 10^{-28}}\ \frac{J}{em^2}\left[2.3 \times 10^{-28}\ \frac{J}{em^2}\right]^{-1} \times \left[\frac{e}{4\pi\varepsilon_0 a_0^3}\right]$$

$$= \frac{-4 \times 6.6 \times 10^{-34} \times 10^{7}}{2 \times 10^{-28}} \; \frac{1}{2.3 \times 10^{-28}} \left[\frac{e}{4\pi\varepsilon_0 a_0^3}\right]$$

$$= \boxed{-5.74 \times 10^{29} \left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)}$$

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